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An elementary new proof of the determination of a convex function by its subdifferential

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Rockafellar proved that any closed, convex function is uniquely determined by its subdifferential mapping up to an additive constant. The aim of this article is to provide an elementary proof of the same result.

Keywords: subdifferential mapping; convex analysis; convex function

AMS Subject Classifications: 49J52; 49J53; 52A41

A well-known theorem of convex analysis states that a convex, closed, proper function f is uniquely determined up to a constant from its subdifferential ∂f [1]. The aim of this note is to provide an elementary proof of a slightly more general result, namely that f is uniquely determined up to a constant if we know one element of $\partial f(x)$ for each x in the relative interior of its domain.

THEOREM 1 *Let f, g be real functions on a convex set C (subset of a vector space) such that*

$$\forall x \in C : \partial f(x) \cap \partial g(x) \neq \emptyset,$$

then $f = g + \text{const}$.

Proof Consider any $x, y \in C$. Choose

$$x^* \in \partial f(x) \cap \partial g(x),$$

$$y^* \in \partial f(y) \cap \partial g(y).$$

For any $m \in \mathbb{N}$, set

$$x_k = x + k \frac{y - x}{m}, \quad k = 0, \dots, m,$$

and choose

$$x_k^* \in \partial f(x_k) \cap \partial g(x_k), \quad k = 1, \dots, m - 1.$$

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Put $x_0^* := x^*$, $x_n^* := y^*$. From the definition of the subdifferential it follows that

$$\begin{aligned} f(x_{k+1}) - f(x_k) &\geq \left\langle x_k^*, \frac{y-x}{m} \right\rangle, \\ g(x_k) - g(x_{k+1}) &\geq -\left\langle x_{k+1}^*, \frac{y-x}{m} \right\rangle, \end{aligned}$$

for all $k=0, \dots, m-1$. Summing up all these inequalities we obtain

$$f(y) - f(x) - g(y) + g(x) \geq \frac{1}{m} \langle x_0^* - x_n^*, y - x \rangle \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Thus

$$f(y) - f(x) \geq g(y) - g(x). \quad (1)$$

By switching f and g we get the inequality (1) in the opposite direction, thus there is an equality in (1) for any $x, y \in C$ and so $f = g + \text{const}$. ■

COROLLARY 1 *If $f, g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ are proper, closed, convex functions with $\text{cl}(\text{dom } \partial f) = \text{cl}(\text{dom } \partial g) =: D$ and*

$$\forall x \in \text{ri} D : \partial f(x) \cap \partial g(x) \neq \emptyset,$$

then $f = g + \text{const}$.

Proof Due to convexity of f , we have

$$\text{ri}(\text{dom } f) \subset \text{dom}(\partial f) \subset \text{dom } f.$$

Since $\text{ri}(\text{dom } f)$ and $\text{dom } f$ have the same affine hull, $\text{dom } \partial f$ has the same affine hull, say K . Hence, we can apply relative interior operation on the inclusions in (2), because the relative interiors are interiors with respect to K . We obtain

$$\text{ri}(\text{dom } \partial f) = \text{ri}(\text{dom } f),$$

thus $\text{ri } D = \text{ri}(\text{dom } f)$ is a convex set.

Applying the theorem for $C := \text{ri} D \neq \emptyset$, we obtain $f = g + c$ ($c \in \mathbb{R}$) on $\text{ri } D$. Choose a point $z \in \text{ri } D$ and consider any point $y \in D$, so the segment $[z, y]$ is a subset of the convex set $\text{ri } D$. Since the function f is closed, we have

$$f(y) \leq \liminf_{t \searrow 0} f((1-t)y + tz),$$

and from the convexity, $f((1-t)y + tz) \leq (1-t)f(y) + tf(z)$, where $f(z)$ is finite, so by letting $t \searrow 0$ we obtain

$$f(y) \geq \limsup_{t \searrow 0} f((1-t)y + tz).$$

Hence $f(y) = \lim_{t \searrow 0} f((1-t)y + tz)$, and similarly $g(y) = \lim_{t \searrow 0} g((1-t)y + tz)$. Thus $f(y) = g(y) + c$, for any $y \in D$, because $f(y)$ and $g(y)$ are both limits of the values of f and g at the points of the segment $[z, y] \subset \text{ri } D$, on which $f = g + c$ holds. Moreover, $f(y) = +\infty = g(y)$ for any $y \notin D$. Indeed, from $\text{ri}(\text{dom } f) \subset \text{dom}(\partial f)$,

we obtain $\text{dom } f \subset \text{cl}(\text{dom } \partial f) = D$. Similarly, $\text{dom } g \subset D$. Consequently $f = g + \text{const}$. ■

The corollary implies that a convex, closed, proper function f is uniquely determined up to a constant if we know $\text{cl}(\text{dom } \partial f)$ and one element of $\partial f(x)$ for each point of $\text{ri}(\text{dom } \partial f)$.

Reference

- [1] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, New Jersey, 1970, pp. 239–240.